UDC 531.36

ON STABLE MOTIONS

PMM Vol. 38, №2, 1974, pp. 366-369
Iu. M.-L. KOSTIUKOVSKII
(Moscow)
(Received January 15, 1973)

We consider a controlled system described by a system of ordinary differential equations the right-hand sides of which contain arbitrary smooth functions of time. Conditions are formulated under which all possible motions of this system do not include a single Liapunov-stable motion belonging to a given bounded set in the space of its states.

1. The motion of the controlled system is defined by the system of equations

$$dx / dt = Q(t, x) + u(t), \qquad t \in I = \{t : t \geqslant t_0\} \tag{1.1}$$
$$x = (x_1, \ldots, x_n),\ u = (u_1, \ldots, u_n),\ Q(\cdot) = Q_1(\cdot), \ldots, Q_n(\cdot))$$

Here x, u and $Q(\cdot)$ are vectors in the real n-dimensional space R_x^n; t_0 is the initial instant and t is time. The vector $x = x(t)$ characterizes the state of the controlled system; $u = u(t)$, $t \in I$ is the control, the mapping of which is $\omega = \{(t, u) : u = u(t), t \in I\}$, $\omega[t_0, t]$ is the contraction of ω onto $[t_0, t] \cap I$; $x(t) = \varphi(t, t_0, x_0, \omega[t_0, t])$, $t \in I$ describes the motion of the system (1.1) for the given $\omega \in \Omega$ originating from the initial state $x(t_0) = x_0$, and Ω is a certain admissible set $\{\omega\}$.

We shall formulate a problem related to the problem first formulated by Chetaev [1, 2] and concerned with separating sets of stable and unstable motions out of a general continuous set of motions.

Problem. Using the form of the system (1.1) which can be identified with the controlled system, we must find the conditions under which all possible motions of the system do not include a single stable motion belonging to the given set G in R_x^n for all $t \in I$. We denote

$$\operatorname{div}_x Q(t, x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} Q_i(t, x), \qquad \|x\| = \max_i |x_i|$$

$$d(a, Z) = \inf \{\|a - b\|, b \in Z\}, \quad S(Z, \rho) = \{x \in R_x^n : d(x, Z) < \rho\}$$

Here Z is any set in R_x^n and $\bar{S}(Z, \rho)$ is the closure of $S(Z, \rho)$, where ρ is a positive number.

Theorem 1. Let (a) G be a given bounded set in R_x^n; (b) $u(t)$, $t \in I$ be any continuously differentiable function of time; (c) a positive number ε_0 exists such that the function $Q(t, x)$ and its partial derivatives in t and $x_1, \ldots, x_n$ are all continuous in $I \times \bar{S}(G, \varepsilon_0)$; (d) a continuous function $M(x)$ exists such that $\operatorname{div}_x Q(t, x) \geqslant M(x) > 0$ for all $t \in I$ and all $x \in \bar{S}(G, \varepsilon_0)$. Then, no matter which control $u(t)$, $t \in I$ is chosen from the class of continuously differentiable functions of time, no Liapunov-stable motion of (1.1) exists belonging to the set G for all $t \in I$.

Proof. Suppose that contrary to the theorem there exists a control $u^*(t)$, $t \in I$ belonging to the class of continuously differentiable functions of time, and a corresponding Liapunov-stable motion $x^0(t) = \varphi(t, t_0, x_0^*, \omega^*[t_0, t])$, $t \in I$, $x_0^* \in G$ such that

$$x^0(t) \in G \quad \text{for all} \quad t \in I \tag{1.2}$$

Consequently, in accordance with the Liapunov's definition of a stable motion [3] there exists a positive number $\delta \leqslant {}^1/_2\ \varepsilon_0$ such that

$$\| \varphi(t, t_0, x_0, \omega^* [t_0, t]) - x^0(t) \| < {}^1/_2\ \varepsilon_0 \tag{1.3}$$

for all $t \in I$ and all $x_0 \in V_0 = \{x_0 : \| x_0 - x_0^* \| < \delta \leqslant {}^1/_2\ \varepsilon_0\}$.

From (1.2) and (1.3) we see that the motions $\Phi = \{\varphi(t, t_0, x_0, \omega^* [t_0, t]),\ t \in I,\ x_0 \in V_0\}$ belong to the set $S(G, {}^1/_2\ \varepsilon_0)$ and are therefore bounded, by the condition (a) of the theorem. On the other hand, since $\bar{S}(G, {}^1/_2\varepsilon_0)$ is a compactum, the condition (d) of the theorem implies that a positive number α exists such that

$$\operatorname{div}_x Q(t, x) \geqslant M(x) \geqslant \alpha > 0 \tag{1.4}$$

for all values of $t \in I$ and all $x \in \bar{S}(G, {}^1/_2\varepsilon_0)$. But in this case from (1.3) and (1.4) it follows that

$$\int_{t_0}^{t} \operatorname{div}_x Q(\tau, \varphi(\tau, t_0, x_0, \omega^* [t_0, \tau])\, d\tau \to \infty \quad \text{when} \quad t \to \infty \tag{1.5}$$

uniformly in $x_0 \in V_0$, where V_0 is an open set in R_x^n. By Theorem 1 of [4] the relation (1.5) means that the motions $\Phi = \{\varphi(t, t_0, x_0, \omega^* [t_0, t]),\ t \in I$ and $x_0 \in V_0\}$ are completely labile relative to the set V_0 in the sense of [4] and this contradicts the fact that the motions Φ are bounded, which was established above.

Note 1. Let $n = 2$, $Q(t, x) \equiv A(x)$, $u(t) \equiv 0$. In this particular case Theorem 1 resembles the known Bendixon criterion [5] of the absence of the limit cycles from the given open region of the phase plane R_x^2. However, unlike the Bendixon criterion, Theorem 1 formulates the sufficient conditions for the absence fo any Liapunov-stable motions (not necessarily periodic) in the given bounded (not necessarily open) set G in R_x^2 provided that the divergence of $A(x)$ is sign positive in $S(G, \varepsilon_0)$ (i.e. has a property stronger than that of the constancy of sign on which the Bendixon criterion [5] depends).

2. If $Q(t, x) \equiv A(x)$ and $u(t) \equiv 0$, then the system (1.1) assumes the form

$$dx / dt = A(x), \qquad t \in I \tag{2.1}$$

Theorem 2. Let: (a) G be a bounded set in R_x^n; (b) a positive number ε_0 exists such that $A(x)$ is a function continuously differentiable in $x_1, \ldots, x_n$ in $S(G, \varepsilon_0)$; (c) a scalar function $\nu(x)$ exists which is continuously differentiable in $x_1, \ldots, x_n$ in $S(G, \varepsilon_0)$ and such that $\nu(x) \neq 0$ and $\operatorname{div}_x [\nu(x) A(x)] > 0$ for all $x \in S(G, \varepsilon_0)$. Then no Liapunov-stable motion of (1.1) exists belonging to the set G for all $t \in I$.

Proof. By assuming that the opposite is true we can establish , as in the proof of Theorem 1, that an open set V_0 of the initial states of the system (2.1) exists such that the motions of this system $\Phi = \{\varphi(t, t_0, x_0),\ t \in I,\ x_0 \in V_0\}$ belong to the region $S(G, {}^1/_2\ \varepsilon_0)$ and are therefore bounded.

Consider the auxilliary system

$$dz / dt = \nu(z) A(z), \quad t \in I, \quad z = (z_1, \ldots, z_n) \in R_x^n \tag{2.2}$$

By the condition (c) of Theorem 2 the function $\nu(z)$ has a constant sign in $S(G, \varepsilon_0)$, therefore the phase trajectories of (2.1) must coincide with the phase trajectories of the auxilliary system (2.2) in $S(G, \varepsilon_0)$. In this case the motions of (2.2) originating at V_0

must belong to the bounded set $S(G, {}^1/_2\varepsilon_0) \subset R_x^n$ just as the motions Φ of the basic system (2.1). But by the conditions (c) of Theorems 1 and 2 this cannot take place.

Note 2. Let $n = 2$. In this particular case Theorem 2 resembles the well known Dulac criterion of the absence of limit cycles from the given open region of the phase plane R_x^2 [6] just as Theorem 1 resembles the Bendixon criterion [5].

3. Let a system be given, consisting of two coupled Van der Pol oscillators. This system can be interpreted as e.g. a mathematical model of a twin-contour self-excited oscillator [7]

$$\begin{aligned} &dx_1 / dt = x_2, \qquad dx_3 / dt = x_4 \\ &dx_2 / dt = -\omega_1^2 x_1 - 2\delta_1 x_2 + (\alpha_1 - \beta_1 x_1^2) x_2 + \gamma_1 x_3 + u_1(t) \quad (t \geqslant 0) \\ &dx_4 / dt = -\omega_2^2 x_3 - 2\delta_2 x_4 + (\alpha_2 - \beta_2 x_3^2) x_4 + \gamma_2 x_1 + u_2(t) \end{aligned} \tag{3.1}$$

Here $x = (x_1, \ldots, x_4)$ is a vector in a real, four-dimensional normed space R_x^4 with the norm $\| x \| = \max_{1, \ldots, 4} | x_i |$; $u_1(t)$, and $u_2(t)$ $(t \geqslant 0)$ are the input variables represented by any continuously differentiable scalar functions of time and $\omega_i, \alpha_i, \beta_i, \gamma_i, \delta_i$ $(i = 1, 2)$ are positive numbers satisfying the condition

$$\alpha_1 + \alpha_2 \leqslant 2(\delta_1 + \delta_2) \tag{3.2}$$

Problem. To find a bounded set G in R_x^4, which contains no Liapunov-stable motions of the system (3.1) belonging to this set, for all $t \geqslant 0$.

Obviously (3.1) satisfies the conditions (b) and (c) of Theorem 1. The set G can be found from the condition (d) of Theorem 1. Using this condition we first find the set $S(G, \varepsilon_0)$ and use this to construct G by choosing a suitable ε_0.

From (3.1) it follows that

$$\mathrm{div}_x Q(t, x) = -\beta_1 x_1^2 - \beta_2 x_2^2 + (\alpha_1 + \alpha_2) - 2(\delta_1 + \delta_2) \tag{3.3}$$

Let E denote a region of R_x^4 in which $\mathrm{div}_x Q(t, x) \equiv M(x) > 0$. Then, taking (3.3) into account, we can write

$$E = \left\{ x = (x_1, \ldots, x_4) : \left(\frac{x_1}{\sqrt{\beta\beta_2}} \right)^2 + \left(\frac{x_3}{\sqrt{\beta\beta_1}} \right)^2 < 1 \right\}$$

$$\beta = [2(\delta_1 + \delta_2) - (\alpha_1 + \alpha_2)]$$

where β is a positive number (by virtue of (3.2)).

Let $\mathrm{Pr}_{x_1 o x_3}[*]$ be the projection of the set «*» in R_x^4 onto the coordinate plane

$$x_1 o x_3 = \{x = (x_1, \ldots, x_4);\ x_2 = 0,\ x_4 = 0\}$$

The condition (d) of Theorem 1 can be guaranteed to hold by taking, as $S(G, \varepsilon_0)$, any open bounded connected set in R_x^4 such, that

$$\mathrm{Pr}_{x_1 o x_3}[S(G, \varepsilon_0)] = \mathrm{Pr}_{x_1 o x_3}[E] = \left\{ (x_1, x_3): \left(\frac{x_1}{\sqrt{\beta\beta_2}} \right)^2 + \left(\frac{x_3}{\sqrt{\beta\beta_1}} \right)^2 < 1 \right\}$$

Below we show that if ε_0 satisfies the conditions

$$0 < \varepsilon_0 < \min(\sqrt{\beta\beta_1};\ \sqrt{\beta\beta_2}) \tag{3.4}$$

then the set G sought can be chosen nonempty.

Let us denote by $\Gamma(*)$ the boundary of the set «*». On the coordinate plane $x_1 o x_3$ we construct the following sets:

$$B(\varepsilon_0) = \{(x_1, x_3) : \max(|x_1 - d_1|; |x_3 - d_3|) \leqslant \varepsilon_0; \quad (d_1, d_3) \in \Gamma(\mathrm{Pr}_{x_1 0 x_3}[E])\}$$

$$K(\varepsilon_0) = B(\varepsilon_0) \cap \mathrm{Pr}_{x_1 0 x_3}[E], \quad U(\varepsilon_0) = \mathrm{Pr}_{x_1 0 x_3}[E] \setminus K(\varepsilon_0)$$

Taking the definition of the norm $\|x\| = \max_{1, \ldots, 4} |x_i|$ into account, we find from the previous constructions, that we can take as G any open bounded connected set in R_x^4 such that

$$\mathrm{Pr}_{x_1 0 x_3}[G] = U(\varepsilon_0)$$

Naturally G is nonempty because ε_0 satisfies the conditions (3.4). Thus all conditions of Theorem 1 hold. Consequently, no matter how small the positive number ε_0 and what continuously differentiable functions $u_1(t)$ and $u_2(t)$ are chosen, no Liapunov-stable motion of (3.1) exists belonging, at all $t \geqslant 0$, to the set G constructed.

The author thanks V. V. Rumiantsev for valuable advice.

REFERENCES

1. Chetaev, N. G., On the stable dynamic trajectories. Sb. nauchn. tr. Kasansk. aviats. inst., № 5, 1936.
2. Chetaev, N. G., On the stable dynamic trajectories. Uch. zap. Kazansk. un-ta, Vol. 91, Book 4, № 1, 1931.
3. Chetaev, N. G., Stability of Motion. (English transaltion), Pergamon Press, Book № 09505, 1961.
4. Kostiukovskii, Iu. M.-L., On an idea of Chetaev. PMM Vol. 37, № 1, 1973.
5. Bendixon, I., Sur les courbes definies par des équations différentielles. Acta Math., Bd. 24, S. 1–88, 1901.
6. Dulac, H., Recherche des cycles limites. Compt. rend., Acad. Sci., t. 204, №23, p. 1703–1706, 1937.
7. Blaquire, A., Nonlinear system analysis. New York - London, Acad. Press, 1966.

Translated by L. K.

UDC 533.6.011.8

ON THE PROBLEM OF STUDYING EXACT SOLUTIONS OF A SYSTEM OF EQUATIONS OF KINETIC MOMENTS

PMM Vol. 38, № 2, 1974, pp. 369-372

V. Ia. RUDIAK
(Novosibirsk)

(Received 21 November 1972)

Galkin in his papers [1, 2] obtained a class of exact solutions of the system of kinetic moments for a monatomic Maxwell-type gas. The simplest flows described by these solutions, namely the shear and divergent flows, were used to analyze the domain of applicability of the Chapman-Enskog method [1, 3, 4]. The present paper deals with certain other flows belonging to this class. The solutions obtained are used to investigate the domain of applicability of the Navier-Stokes and Barnett approximations to the Chapman-Enskog method.

1. Let us consider a one-dimensional flow for which the macroscopic velocity com-